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Higher-order Noether symmetries and constants of the motion

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Abstract. In this paper we identify a class of symmetries for Lagrangian systems, called higher-order Noether symmetries, which yield a corresponding first integral without further integrations. First, a version of Noether's theorem is recalled in which Noether symmetries are considered to be symmetries of the two-form $d\theta$, θ being the Cartan form. nth-order Noether symmetries are then defined by the requirement $L_Y^n \, d\theta = 0$. The picture is further generalised by exploring various ways in which the computation of repeated Lie derivatives of $d\theta$ can lead to the identification of first integrals. A number of illustrative examples are discussed.

1. Introduction

The problem of relating constants of the motion to symmetries has always found considerable interest in various branches of theoretical physics. For classical Lagrangian mechanics and classical field theory, there is no doubt that Noether's theorem (Noether 1918) provides the most celebrated result in this matter. Although it has never really been absent in subsequent scientific literature, one can easily say that this theorem and various generalisations have gained renewed interest during the last decade. While on the one hand it is instructive to see how many authors use different approaches, it is rather unfortunate and confusing that these approaches often show important aberrations in their conclusions, and are therefore of a rather controversial nature.

In a recent paper (Sarlet and Cantrijn 1981), we have made a comparative study of different approaches to Noether's theorem in classical mechanics, in an attempt to clarify conceptually the origin of these differences and present arguments in favour of a best possible approach. We refer to this paper for more references on the subject. Let us, however, briefly summarise some of the achievements of this study. In the first place, in any theory aimed at establishing a link between symmetries and conservation laws, we favour those approaches which succeed in showing a kind of exclusiveness in this relationship. From this point of view one can criticise generalisations of Noether's theorem which go so far that all symmetries correspond to all constants of the motion, as is the case, for example, in treatments by Rosen (1972, 1974) and Candotti *et al* (1972). Note that similar criticism was formulated by Martinez Alonso (1979). Our study further revealed that the kind of restrictions on the gauge function, proposed e.g. by

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Lutzky (1979), is not very appropriate for achieving a form of one-to-one correspondence between symmetries and first integrals. Most importantly, we showed that the two main approaches to Noether's theorem for velocity-dependent symmetry generators do have that kind of exclusiveness, and that moreover they are essentially equivalent. One of these approaches is based on classical calculus of variations in which infinitesimal transformations are required to leave the action integral invariant up to gauge terms. As a typical paper of this nature, we can cite a treatment by Djukic (1973), which was recently generalised by Djukic and Strauss (1980) to systems described by Lagrangian functions depending on second-order derivatives and with non-conservative forces. The other approach makes use of the characterisation of Lagrangian systems in terms of vector fields and differential forms on the tangent bundle of a manifold. A good reference here is a paper by Crampin (1977). In that context we defined Noether transformations to be symmetries of the two-form $d\theta$, θ being the so-called Cartan form $p_i\, dq^i - H\, dt$, expressed in Lagrangian coordinates.

The higher-order Noether symmetries, which we intend to discuss in the present paper, find their origin in a generalisation of this second approach.

If for a moment we go back to a more general framework, one of the first questions which arises in a natural way is this: if the given second-order system is not of Lagrangian type, could there still be a way to associate a first integral with every symmetry, and similarly even for a Lagrangian system, if the symmetry is not of Noether type, can it be associated with a constant of the motion? Such questions have been discussed e.g. by Lutzky (1979), Prince and Eliezer (1980) and Prince (1980). These authors consider a first integral as being associated with a given symmetry if it is in addition an invariant of the symmetry vector field itself. Unlike with Noether's theorem, however, the determination of a constant obeying this rule still requires the solution of a system of differential equations. It is interesting to note that the class of higher-order Noether symmetries which we will introduce here directly yields a corresponding first integral with the above property, without further integrations.

The plan of the paper is as follows. In § 2 we briefly recall the definition of a Lagrangian system via a characteristic vector field of $d\theta$, and the formulation of Noether's theorem via symmetries of $d\theta$. In § 3 we proceed to the derivation of higher-order Noether symmetries and their related constant of the motion. Section 4 gives an analysis of a more general structure behind these results, which will help us to understand them better. In § 5 we present a number of examples, in which use is made of a class of symmetries for linear systems, derived in an Appendix. Section 6 contains some general comments. For some basic concepts (and properties) of differential geometry, which are used throughout this paper, the reader may consult e.g. Godbillon (1969) or Hermann (1968).

2. Lagrangian systems and Noether's theorem

Consider a configuration manifold M and the associated extended tangent bundle $\mathbb{R}\times TM$, on which we choose a set of natural local coordinates $(t, q^i, \dot{q}^i)$ $(i = 1, \ldots, n)$. Let L be a function on $\mathbb{R}\times TM$, which is regular in the domain of the coordinate chart, in the sense that the Hessian matrix $\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j$ is non-singular. Consider the one-form

$$\theta = L\, dt + (\partial L/\partial \dot{q}^i)(dq^i - \dot{q}^i\, dt). \tag{1}$$

We say that a vector field Γ defines (locally) a Lagrangian system (with Lagrangian L), if it has time-component one, and is a characteristic vector field of $d\theta$, i.e. we have

$$i_\Gamma \, d\theta = 0, \qquad \langle \Gamma, dt \rangle = 1. \tag{2}$$

It is straightforward to check that the requirements (2) will imply that Γ is of the form

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \Lambda^i(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} \tag{3}$$

and that moreover the system of second-order differential equations associated with Γ, namely

$$\ddot{q}^i = \Lambda^i(t, q, \dot{q}), \tag{4}$$

will have the familiar structure of a system of Euler–Lagrange equations, but written in normal form. Next, consider a general vector field Y on $\mathbb{R} \times TM$, with coordinate representation

$$Y = \tau(t, q, \dot{q}) \frac{\partial}{\partial t} + \xi^i(t, q, \dot{q}) \frac{\partial}{\partial q^i} + \eta^i(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}. \tag{5}$$

Y is a symmetry of Γ if $[Y, \Gamma] = 0$.

Infinitesimal transformations generated by such a symmetry leave the differential equations (4) invariant. If, with regard to this invariance, one allows a change of the parametrisation of (4), the bracket of Y and Γ need not vanish. Instead, one obtains the requirement that

$$[Y, \Gamma] = g\Gamma \tag{6}$$

for some function g (see Crampin (1977) or Hermann (1968 p 42)). We will refer to vector fields with this property as general 'dynamical symmetries' of Γ. Note that, in view of the specific structure (3) of Γ, the requirement (6) explicitly reads

$$(\eta^i - \Gamma(\xi^i)) \frac{\partial}{\partial q^i} + (Y(\Lambda^i) - \Gamma(\eta^i)) \frac{\partial}{\partial \dot{q}^i} - \Gamma(\tau) \frac{\partial}{\partial t} = g\Gamma$$

from which it follows that

$$g = -\Gamma(\tau), \tag{7}$$

$$\eta^i = \Gamma(\xi^i) - \dot{q}^i \Gamma(\tau). \tag{8}$$

It can be shown that symmetries of the two-form $d\theta$ constitute a particular class of dynamical symmetries. It is this class which is used in our formulation of Noether's theorem.

Definition. A Noether symmetry of the Lagrangian system Γ is a symmetry of the two-form $d\theta$, i.e. a vector field Y with the property

$$L_Y \, d\theta = 0. \tag{9}$$

Note that, in this local description, a requirement like (9) implies

$$L_Y \theta = df \tag{10}$$

for some 'gauge function' f, and we obtain the following results.

Theorem 1.

(i) To each Noether symmetry Y corresponds a constant of the motion

$$F = f - \langle Y, \theta \rangle \tag{11}$$

which is unique up to a trivial constant.

(ii) To each first integral F corresponds a Noether symmetry Y, which is unique up to a trivial dynamical symmetry $h\Gamma$.

(iii) F is in addition an invariant of Y, i.e. we have

$$Y(F) = 0. \tag{12}$$

For more details about these matters, we refer to our earlier quoted review paper and references therein.

3. Higher-order Noether symmetries and constants of the motion

Consider a Lagrangian system Γ as defined by (2). Let Y be a dynamical symmetry of Γ, and define the one-form α by the inner product of Y with the two-form $d\theta$. This means that basically the quantities Γ, Y, θ, and α are interrelated by the formulae

$$i_\Gamma \, d\theta = 0, \tag{13a}$$

$$[Y, \Gamma] = g\Gamma, \tag{13b}$$

$$i_Y \, d\theta = \alpha. \tag{13c}$$

Properties of the one-form α

(i) From (13a, c) we trivially obtain

$$\langle \Gamma, \alpha \rangle = 0, \tag{14a}$$

$$\langle Y, \alpha \rangle = 0. \tag{14b}$$

(ii) α is an invariant one-form under Γ. Indeed, we have

$$\begin{aligned} L_\Gamma \alpha &= L_\Gamma i_Y \, d\theta \\ &= i_Y L_\Gamma \, d\theta - i_{[Y,\Gamma]} \, d\theta = 0, \end{aligned} \tag{15}$$

in view of (13a, b).

If α happens to be exact ($\alpha = dF$), Y is a Noether symmetry, and we are in the situation covered by theorem 1. If α is not exact, no direct formula exists for relating the symmetry Y to a first integral F. However, we can make, so to speak, a second try. Indeed, in view of (14a), we have

$$L_\Gamma \alpha = i_\Gamma \, d\alpha.$$

Therefore, by (15) we can write

$$i_\Gamma \, d\alpha = 0 \tag{16a}$$

and also

$$[Y, \Gamma] = g\Gamma, \tag{16b}$$

$$i_Y \, \mathrm{d}\alpha = \beta, \tag{16c}$$

where the latter relation defines the one-form β.

Comparing (16) with (13), we see that we are back in the original situation but with α replacing θ. Of course, the new relations (16) are only of interest if $\mathrm{d}\alpha$ is not a constant multiple of $\mathrm{d}\theta$. Under this assumption it could happen that this time β is exact, say

$$\beta = \mathrm{d}F. \tag{17}$$

In that case it immediately follows from properties (i) written for β that F is a constant of the motion of Γ, and is in addition an invariant of the symmetry Y. The necessary and sufficient condition for β to be exact is easily obtained as follows:

$$0 = \mathrm{d}\beta = L_Y \, \mathrm{d}\alpha = L_Y L_Y \, \mathrm{d}\theta. \tag{18}$$

In comparison with the definition of a Noether symmetry (9), we call a dynamical symmetry satisfying (18) a 'second-order Noether symmetry'.

When β is not exact, it is clear that we can repeat the whole story as many times as wanted. In each step we will recover a set of relations of type (13) or (16), with the possibility that the newly obtained one-forms α, β, γ, ... could be exact and lead to a corresponding first integral. We can summarise this whole procedure as follows.

Definition. A dynamical symmetry Y of Γ is called a Noether symmetry of order n if

$$L_Y^n \, \mathrm{d}\theta = 0 \tag{19}$$

and $L_Y^k \, \mathrm{d}\theta \neq 0$ for $k < n$.

Theorem 2.

(i) To each Noether symmetry of order n ($\geqslant 1$) corresponds a constant of the motion F, determined by

$$L_Y^{n-1} i_Y \, \mathrm{d}\theta = \mathrm{d}F. \tag{20}$$

(ii) F has the invariance property $Y(F) = 0$.

Proof. From the assumptions (6) and (19) it follows that:

(1) $$\mathrm{d}L_Y^{n-1} i_Y \, \mathrm{d}\theta = L_Y^{n-1} \, \mathrm{d}i_Y \, \mathrm{d}\theta = L_Y^n \, \mathrm{d}\theta = 0,$$

hence $L_Y^{n-1} i_Y \, \mathrm{d}\theta = \mathrm{d}F$ for some F;

(2) $$\begin{aligned} i_\Gamma L_Y^{n-1} i_Y \, \mathrm{d}\theta &= L_Y i_\Gamma L_Y^{n-2} i_Y \, \mathrm{d}\theta + i_{[\Gamma, Y]} L_Y^{n-2} i_Y \, \mathrm{d}\theta \\ &= (L_Y - g) i_\Gamma L_Y^{n-2} i_Y \, \mathrm{d}\theta \\ &= (L_Y - g)^{n-1} i_\Gamma i_Y \, \mathrm{d}\theta = 0, \end{aligned}$$

hence $\Gamma(F) = 0$;

(3) $$i_Y L_Y^{n-1} i_Y \, \mathrm{d}\theta = L_Y^{n-1} i_Y i_Y \, \mathrm{d}\theta = 0,$$

hence $Y(F) = 0$.

Note that the analogue of formula (11) for the Noether constant here becomes

$$F = f - L_Y^{n-1} \langle Y, \theta \rangle, \tag{21}$$

with $\mathrm{d}f = L_Y^n \theta$.

Some comments are in order now. First of all, although the case $n = 1$ (i.e. Noether's theorem) appears here merely as a particular case of the above results, it must be said that there remains a clear distinction between $n = 1$ and $n > 1$. This distinction already emerges in the definition. Indeed, a vector field Y satisfying (19) for $n = 1$ is automatically a dynamical symmetry of Γ. This is no longer the case for $n > 1$, so that the symmetry requirement was explicitly stated in addition to condition (19). In comparing theorems 1 and 2, one also notices that there is no converse statement in theorem 2, which is, however, not a drawback. Theorem 2 must be seen as providing a possible way of deriving a first integral if a dynamical symmetry is known which is not a Noether symmetry. If, conversely, a first integral is known, it would not make much sense to relate it to a higher-order Noether symmetry, if it can already be understood as being generated by a simple Noether symmetry, according to the converse statement in theorem 1.

Secondly, as stated in the Introduction, the preceding results are not at all restricted to Lagrangian mechanics. As a matter of fact, it is quite easy to identify the essential features we need for deriving these results, even in a global way. Consider a general manifold N, on which an exact two-form $\mathrm{d}\theta$ is globally defined (not necessarily of maximal rank). Let a dynamical system on N be defined by a vector field Γ which is characteristic for $\mathrm{d}\theta$, so that equation (13a) holds. If in addition we assume that N is simply connected, all the above results hold globally. Indeed, under these circumstances the converse of the Poincaré lemma holds globally for closed one-forms, which is essentially the only tool we needed. For manifolds which are not simply connected, we get at least local first integrals.

With this in mind, we intend in the next section to generalise a bit further the results contained in theorem 2. Now it certainly is possible to construct a much more abstract general theory, from which theorem 2 would follow as a special case. However, it is clear, from the terminology we have introduced, that in the first place we want to stay close to Noether's theorem in Lagrangian mechanics. Therefore we will limit ourselves to some simple considerations which will help to understand in what sense an occurrence like (19) can be situated within relations which will always hold, when one computes subsequent powers of the Lie derivative, applied to $\mathrm{d}\theta$.

4. Some generalisations

It is worthwhile considering first the following remark. Our original idea, as explained in the previous section, consisted in trying to detect exact invariant one-forms (like α, β, etc) by consecutively deriving sets of relations like (13) and (16). Now the condition that $L_Y^k\,\mathrm{d}\theta = 0$ for some k is not the only way in which such an invariant exact one-form can be discovered. Indeed, if the kth-order Lie derivative of $\mathrm{d}\theta$ is a linear combination of lower-order derivatives with constant coefficients, a similar linear combination of the forms $\alpha, \beta, \ldots$ will be exact, and invariant under Γ. As an example, consider the case that

$$L_Y^2\,\mathrm{d}\theta = cL_Y\,\mathrm{d}\theta \qquad (c \text{ constant}). \tag{22}$$

From the definition of α and β, (22) implies

$$\mathrm{d}\beta - c\,\mathrm{d}\alpha = 0,$$

hence (locally)

$$\beta - c\alpha = dF,$$

which in view of properties (i) for α and β will imply that F is a constant of Γ and Y.

It could be said that for a given dynamical symmetry Y of a Lagrangian system Γ, a relation like (19) or (22) will only accidentally be true. Nevertheless we would like to show that it can in any case be useful to compute consecutive Lie derivatives of $d\theta$, once a symmetry Y is known. At least locally, since the space of two-forms is finite-dimensional, after a sufficient number of derivatives we must get a relation of the type

$$L_Y^k \, d\theta = \sum_{l=0}^{k-1} f_l L_Y^l \, d\theta, \tag{23}$$

where the f_l are functions.

The maximal number of derivatives one must consider is even a bit less than the dimension of two-forms. In fact, since $i_\Gamma \, d\theta = 0$ and Y is a dynamical symmetry of Γ, we will have $i_\Gamma(L_Y^m \, d\theta) = 0$ for all m. Hence $d\theta$, $L_Y \, d\theta$, $L_Y^2 \, d\theta, \ldots$ will all be exact absolute integral invariants of order two of the vector field Γ (see e.g. Godbillon 1969). In this way one can show that a relation of type (23) will occur after at most $k = C_{2n}^2$ derivatives (the dimension of our manifold being $2n+1$ here). The relevance of obtaining a relation like (23) is then expressed by the following result.

Theorem 3. Let Y be a dynamical symmetry of Γ, and assume that k is the first number for which (in some local neighbourhood) equation (23) holds true. Then the functions f_l are constants of the motion of Γ.

Proof. We compute the Lie derivative with respect to Γ of both sides of (23), and try to commute L_Γ with all L_Y. Using (6), it is easy to see that all expressions of the form $L_\Gamma L_Y^m \, d\theta$ will eventually be reduced to sums of terms like $L_Y^p L_{f\Gamma} \, d\theta$, which are all zero. As a result we will end up with the relation

$$\sum_{l=0}^{k-1} \Gamma(f_l) L_Y^l \, d\theta = 0.$$

Since we assumed that all derivatives of order lower than k were independent, this will imply $\Gamma(f_l) = 0$, which completes the proof.

It is questionable whether one should say that constants f_l thus obtained are 'generated' by the symmetry Y, since for instance they do not have in general the property $Y(f_l) = 0$. Nevertheless we see that computing higher-order Lie derivatives of $d\theta$ will 'very often' provide us with some constants of the motion. We can describe in precise terms what is meant by 'very often' in the following summary.

Consider again the assumptions of theorem 3. Then we can distinguish between the following situations.

(i) The f_l are not all trivial constants. In that case the non-trivial ones provide us with constants of the motion of Γ which are, however, not necessarily invariant under the flow of Y.

(ii) All f_l are trivial constants c_l. Then we must make a further distinction.

(a) $c_0 \equiv 0$. In this case one can construct a constant of the motion F with property $Y(F) = 0$, as explained with the example (22). If all c_l ($l = 0, \ldots, k-1$) are zero, we of course recover here the higher-order Noether symmetries discussed in the previous section.

(b) $c_0 \neq 0$. In this case there seems to be no way for finding a related first integral without further integrations. Note hereby that it is in principle always possible to find a first integral of Noether type (11) from a general dynamical symmetry (i.e. not necessarily a Noether symmetry). Indeed, it suffices to find a particular solution f of the partial differential equation

$$\Gamma(f) = Y(L) + \Gamma(\tau)L, \tag{24}$$

which follows from requiring that $F = f - \langle Y, \theta \rangle$ be constant. This is of course not a direct construction of a first integral. Moreover, it can be shown that for the present case the situation is even worse in the following sense: solving equation (24) amounts to finding a first integral, because a particular solution for f is immediately at hand, but unfortunately leads through (11) to the zero constant.

With an eye on the search for possible examples, we end this discussion by the following remark. For Lagrangian systems with one degree of freedom ($n = 1$), we obtain $k = 1$ as the maximal number of derivatives one must compute, before obtaining an identity like (23). In other words we will always have

$$L_Y \, \mathrm{d}\theta = f \, \mathrm{d}\theta \tag{25}$$

for some f (possibly zero), so that no cases of higher-order Noether symmetry can be found for these one-dimensional systems.

5. Illustrative examples

As a necessary prerequisite for computing first integrals from the results of theorem 2, we must of course know dynamical symmetries of the given system. Searching for such symmetries amounts to solving a system of partial differential equations (see the Appendix). These equations are greatly simplified if one restricts attention to generators Y which project onto a vector field on $\mathbb{R} \times M$, in other words if the infinitesimal transformations are restricted to be pure coordinate transformations (not depending on velocities). Within this class of symmetries for Lagrangian systems, some will be of Noether type, and others will not. In the examples, we will therefore focus on the non-Noether symmetries of coordinate type, and see whether some of these happen to be higher-order Noether symmetries.

5.1. Harmonic oscillator

A great number of recent papers have in some way or another dealt with the complete eight-parameter symmetry group for the harmonic oscillator. We can refer e.g. to Wulfman and Wybourne (1976), Lutzky (1978) and Leach (1980). It is known that for this system, governed by

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2),$$

five of the symmetry generators are of Noether type. The three non-Noether symmetries are given by (we only write down their projection $Y^{(0)}$ on $\mathbb{R} \times M$)

$$Y_1^{(0)} = q \, \partial/\partial q,$$

$$Y_2^{(0)} = q \sin t\, \partial/\partial t + q^2 \cos t\, \partial/\partial q,$$
$$Y_3^{(0)} = q \cos t\, \partial/\partial t - q^2 \sin t\, \partial/\partial q.$$

Since the system is one-dimensional we cannot expect to find higher-order Noether symmetries among these. It is straightforward to check that we get

$$L_{Y_1}\, \mathrm{d}\theta = 2\, \mathrm{d}\theta,$$
$$L_{Y_2}\, \mathrm{d}\theta = 3(q \cos t - \dot{q} \sin t)\, \mathrm{d}\theta,$$
$$L_{Y_3}\, \mathrm{d}\theta = -3(q \sin t + \dot{q} \cos t)\, \mathrm{d}\theta,$$

which is in agreement with (25), and where indeed $q \cos t - \dot{q} \sin t$ and $q \sin t + \dot{q} \cos t$ are constants of the motion, as predicted by theorem 3.

5.2. *Kepler problem*

For the plane motion of the Kepler problem we have the Lagrangian

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \mu(q_1^2 + q_2^2)^{-1/2}.$$

Prince and Eliezer (1979) recently derived a point-transformation symmetry

$$Y^{(0)} = t\, \partial/\partial t + \tfrac{2}{3} q_1\, \partial/\partial q_1 + \tfrac{2}{3} q_2\, \partial/\partial q_2,$$

which is not a Noether symmetry, but nevertheless can be said to generate the components of the Runge–Lenz vector as invariants by the requirement $Y(F) = 0$. One might expect therefore that we are facing here a higher-order Noether symmetry, or at least a symmetry belonging to the class (iia) of the summary in § 4. Surprisingly, however, this is not the case. As a matter of fact we have

$$L_Y\, \mathrm{d}\theta = \tfrac{1}{3}\, \mathrm{d}\theta$$

(even $L_Y\theta = \frac{1}{3}\theta$), which is the type of 'desperate situation' discussed in (iib). As a result we are unable to find a substitute here for the integration procedure used by Prince and Eliezer.

5.3. *Linear Lagrangian systems*

If we want to find some examples of second-order Noether symmetries, we better take multidimensional systems, which of course significantly enhances the amount of calculations. We therefore only discuss a couple of linear second-order systems, which are taken from a study on the inverse Lagrangian problem by Sarlet (1980). In the Appendix we derive a general class of symmetries for linear systems with constant coefficients, which will be useful for the next examples.

Consider the system

$$\begin{aligned} &\ddot{q}_1 + 2\lambda \dot{q}_1 + 2\dot{q}_2 + bq_1 + cq_2 = 0, \\ &\ddot{q}_2 + 2\lambda \dot{q}_2 + bq_2 = 0, \end{aligned} \tag{26}$$

with Lagrangian

$$L = \mathrm{e}^{2\lambda t}[\dot{q}_1\dot{q}_2 + t\dot{q}_2^2 - bq_1q_2 - (\tfrac{1}{2}c + bt)q_2^2]. \tag{27}$$

It is a system of the type (A2) with $[A, B] = 0$. If we assume that $c \neq 2\lambda$, the general solution of a matrix R satisfying (A9) and (A10) is given by

$$R=\begin{pmatrix}\mu & \nu\\ 0 & \mu\end{pmatrix}, \tag{28}$$

and determines a dynamical symmetry via (A8). This will be a Noether symmetry if $i_Y\,\mathrm{d}\theta$ is exact, where we recall that Y is related to $Y^{(0)}$ by

$$Y=Y^{(0)}+\eta^i\,\partial/\partial\dot{q}^i, \tag{29}$$

with η^i determined by (8). From a straightforward calculation, it is found that $i_Y\,\mathrm{d}\theta=\mathrm{d}F$ if and only if

$$\mu=-\lambda, \qquad \nu=-1. \tag{30}$$

The corresponding first integral is given by

$$F=\mathrm{e}^{2\lambda t}[bq_1q_2+\tfrac{1}{2}(c+2bt)q_2^2+\lambda q_1\dot{q}_2+\lambda\dot{q}_1q_2+(2\lambda t+1)q_2\dot{q}_2+\dot{q}_1\dot{q}_2+t\dot{q}_2^2]. \tag{31}$$

Now Y will be a second-order Noether symmetry if

$$L_Yi_Y\,\mathrm{d}\theta=\mathrm{d}G \tag{32}$$

for some G. This again leads to the requirement $\mu=-\lambda$, but ν this time is arbitrary, and the corresponding constant of the motion G reads

$$G=(\nu+1)\,\mathrm{e}^{2\lambda t}(\dot{q}_2^2+2\lambda q_2\dot{q}_2+bq_2^2), \tag{33}$$

which is of course an expected integral related to the independent equation in q_2.

As a final example, consider the system

$$\begin{aligned}&\ddot{q}_1+2\lambda\dot{q}_1+2\dot{q}_2+2q_1+q_3=0,\\&\ddot{q}_2+2\lambda\dot{q}_2+q_2=0,\\&\ddot{q}_3+2\lambda\dot{q}_3-q_1=0,\end{aligned} \tag{34}$$

which has a Lagrangian representation with

$$\begin{aligned}L=-\mathrm{e}^{2\lambda t}[&\tfrac{1}{2}(\dot{q}_1^2+\dot{q}_3^2)-\tfrac{1}{2}t(4\lambda-t)\dot{q}_2^2+(t-2\lambda)\dot{q}_1\dot{q}_2+\dot{q}_1\dot{q}_3+t\dot{q}_2\dot{q}_3-q_1\dot{q}_2\\&+q_2\dot{q}_3-\tfrac{1}{2}(q_1^2+q_3^2)+\tfrac{1}{2}t(4\lambda-t)q_2^2-(t-2\lambda)(q_1q_2+q_2q_3)-q_1q_3].\end{aligned} \tag{35}$$

For this example we have $[A,B]\neq 0$, but the condition (A14) is satisfied (see Sarlet 1980). As a result, for finding symmetries of type (A8) a matrix R_0 must be obtained satisfying two commutativity conditions (see Appendix). With the resulting symmetries, one can again test the exactness of $i_Y\,\mathrm{d}\theta$ and $L_Yi_Y\,\mathrm{d}\theta$. The calculations are straightforward but very tedious. One obtains the following two generators of second-order Noether symmetries:

$$\begin{aligned}Y_1^{(0)}=&\frac{\partial}{\partial t}+\frac{1}{4\lambda}[(1-4\lambda^2)q_1+(t+4\lambda b)q_2+q_3]\frac{\partial}{\partial q_1}-\lambda q_2\frac{\partial}{\partial q_2}\\&-\frac{1}{4\lambda}[q_1+(t+2\lambda+4\lambda b)q_2+(1+4\lambda^2)q_3]\frac{\partial}{\partial q_3},\end{aligned} \tag{36}$$

$$Y_2^{(0)} = \frac{\partial}{\partial t} + \frac{1}{2\lambda}[(1-2\lambda^2)q_1 + (t+2\lambda b)q_2 + q_3]\frac{\partial}{\partial q_1} - \lambda q_2 \frac{\partial}{\partial q_2}$$
$$-\frac{1}{2\lambda}[q_1 + (t+2\lambda+2\lambda b)q_2 + (1+2\lambda^2)q_3]\frac{\partial}{\partial q_3}. \tag{37}$$

The constant corresponding to Y_1 is found to be essentially the damped oscillator constant for the equation in q_2, namely

$$F = 2\lambda(b+1)F_1$$

where

$$F_1 = e^{2\lambda t}(\dot{q}_2^2 + 2\lambda q_2\dot{q}_2 + q_2^2). \tag{38}$$

Putting

$$G_1 = [\lambda(q_1\dot{q}_2 + q_2\dot{q}_1 + q_2\dot{q}_3 + q_3\dot{q}_2) + q_2\dot{q}_2 + \dot{q}_1\dot{q}_2 + \dot{q}_2\dot{q}_3 + q_1q_2 + q_2q_3]\, e^{2\lambda t},$$
$$G_2 = [\tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_3^2 + q_1^2 + q_3^2) + \lambda(q_1+q_3)(\dot{q}_1+\dot{q}_3) - 2\lambda\dot{q}_1\dot{q}_2 + \dot{q}_1\dot{q}_3 + \tfrac{1}{2}(1-4\lambda^2)(q_2\dot{q}_1 + q_1\dot{q}_2)$$
$$+\tfrac{1}{2}(q_2\dot{q}_3 + q_3\dot{q}_2) - 2\lambda q_2\dot{q}_2 - 3\lambda q_1q_2 + q_1q_3 - \lambda q_2q_3]\, e^{2\lambda t},$$

the constant corresponding to Y_2 has the following explicit form:

$$F' = 2\lambda(b+1)F_1 + F_2, \qquad \text{with } F_2 = tF_1 + G_1. \tag{39}$$

Finally, one can check that Y_1 becomes a Noether symmetry if $b=-1$. The corresponding Noether constant becomes

$$F_3 = -\tfrac{1}{2}t^2F_1 + 2\lambda tF_1 - G_1t - G_2. \tag{40}$$

Note that, for this example, the integrals obtained through second-order Noether symmetries were in fact already derivable from the Noether constant (40). Indeed, in view of the structure of the functions F_1, G_1 and G_2, one can show that F_3 being a constant of the motion necessarily implies that F_1 and F_2 must be constants too.

6. Concluding remarks

It is clear from the above examples that the calculations involved in the search for higher-order Noether symmetries and their related first integrals can be very lengthy. In writing this paper, it was not our intention to promote higher-order Noether symmetries as *the* general method in the search for first integrals of Lagrangian systems. Certain *ad hoc* constructions can sometimes be easier for that goal (see e.g. Sarlet and Bahar 1980). Nevertheless, it is a fact that all methods with a certain degree of generality become computationally complicated when applied to multidimensional systems. Therefore the ideas explored in this paper certainly can have a practical interest, especially in the following sense: if in some way or another a dynamical symmetry of a given Lagrangian (or Hamiltonian) system has been found, it can be worthwhile to compute subsequent Lie derivatives of $d\theta$, because there are several ways in which this can lead to the identification of first integrals (see the summary of § 4). We even suspect that if such a computation would give no results, this will be seen after a few steps, as was the case for the Kepler problem discussed above.

On the theoretical side, we have found it instructive to discover a way for looking at Noether's theorem from a slightly more general point of view, which could be expressed as follows: if we have a dynamical symmetry at our disposal, we can, so to speak, replace the given data (13) step by step by similar new data like (16), where α, β, ... are always invariant one-forms with respect to Γ. Noether's theorem then corresponds to the case that in the original situation α already happens to be an exact one-form.

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Appendix. A class of symmetries for linear systems

A general dynamical symmetry Y for a second-order system (4) is defined by (6). Apart from the relations (7) and (8), the requirement (6) also yields the condition (see e.g. Sarlet and Cantrijn 1981)

$$\Gamma(\eta^i) - \Lambda^i \Gamma(\tau) - Y(\Lambda^i) = 0. \tag{A1}$$

When the η^i are substituted from (8), (A1) gives rise to a system of partial differential equations for ξ^i and τ. For given functions Λ^i, and when the symmetry is required to generate a point transformation (τ and ξ^i functions of t and q only), this system will usually split into a number of simpler equations by the identification of coefficients of equal powers of $\dot{q}$. Reference to (A1) in the subsequent developments must be understood as reference to this whole procedure.

Specifically, we want to apply this scheme to linear systems,

$$\ddot{q} + 2A\dot{q} + Bq = 0, \tag{A2}$$

where A and B are constant $n \times n$ matrices. From the third-order terms in $\dot{q}$, we then learn that τ must be linear in q, say

$$\tau = \alpha(t)^{\mathrm{T}} q + \tau_0(t) \tag{A3}$$

with $\alpha(t)$ an $n \times 1$ matrix, and T standing for transpose. Equating in (A1) coefficients of the quadratic terms in $\dot{q}$ (after appropriate symmetrisation) then shows that the ξ^i can at most be quadratic in q, say

$$\xi^i = \tfrac{1}{2} q^{\mathrm{T}} Q^i(t) q + \rho^i(t)^{\mathrm{T}} q + \xi_0^i(t), \tag{A4}$$

the $Q^i(t)$ being symmetric matrices, and the $\rho^i(t)$ column vectors.

Since a complete analysis of all symmetries would go far beyond our present needs, we immediately consider the simplification $\alpha \equiv 0$, which can be shown to lead to $Q^i \equiv 0$. Introducing the $n \times n$ matrix $R(t)$ by

$$R_{ij} = (\rho^i)_j,$$

the coefficients of the linear terms in $\dot{q}$ give rise to the condition

$$2\dot{R} - 2RA + 2AR = \ddot{\tau}_0 1 - 2\dot{\tau}_0 A \tag{A5}$$

where 1 is the $n \times n$ unit matrix. Finally, the terms arising from (A1) which are independent of $\dot{q}$ lead to two more conditions, namely

$$\ddot{R}-RB+BR=-2A\dot{R}-2\dot{\tau}_0B \tag{A6}$$

plus the requirement that $\xi_0(t)$ must be a solution of the given system (A2).

It is worth noting that equations (A5) and (A6) remain valid without the assumption $\alpha=0$, but then of course they must be supplemented by a number of other complicated equations involving α and Q. Taking the derivative of (A5), it can be shown by some straightforward manipulations that equation (A6) can be replaced by the simpler condition

$$R(B-A^2)-(B-A^2)R=2\dot{\tau}_0(B-A^2)+\tfrac{1}{2}\dddot{\tau}_0 1. \tag{A7}$$

It is clear now that it will be appropriate to consider a further simplification, which is the assumption that τ_0 is constant, say $\tau_0=1$. Choosing moreover $\xi_0^i=0$ we obtain the following result.

Lemma. Consider a linear system of second-order equations (A2). Then the vector field

$$Y^{(0)}=\partial/\partial t+R_{ij}(t)q^j\,\partial/\partial q^i \tag{A8}$$

generates a dynamical symmetry of (A2) if and only if the matrix $R(t)$ satisfies the conditions

$$\dot{R}=RA-AR, \tag{A9}$$

$$R(B-A^2)=(B-A^2)R. \tag{A10}$$

Clearly the system (A2) need not be of Lagrangian type for the validity of this lemma. The fact that (A2) is Lagrangian can, however, simplify the search for solutions of (A9) and (A10). The general solution of (A9) is given by

$$R=\exp(-At)R_0\exp(At), \qquad R_0=R(0). \tag{A11}$$

Introducing the matrix $Z(t)$ by

$$Z(t)=\exp(At)(B-A^2)\exp(-At), \tag{A12}$$

it then follows that (A10) requires R_0 to commute with $Z(t)$. From a Taylor expansion of this condition, one obtains

$$\begin{aligned}&R_0(B-A^2)=(B-A^2)R_0,\\&R_0[A,B]=[A,B]R_0,\\&R_0[A,[A,B]]=[A,[A,B]]R_0,\\&\quad\vdots\end{aligned} \tag{A13}$$

where $[A,B]=AB-BA$.

The case $[A,B]=0$ is particularly interesting, because it is sufficient for the existence of a Lagrangian. It is clearly also sufficient for the existence of a symmetry generator (A8), since all conditions (A13) are identically satisfied, except for the first one. Constructing a constant matrix R_0 commuting with $B-A^2$ will, through (A11), lead to a symmetry of type (A8). If, for instance, we have $[A,B]\neq 0$, but

$$[A,[A,B]]=0, \tag{A14}$$

we find a symmetry (A8) if an R_0 exists which commutes with both $B-A^2$ and $[A,B]$.

It would be interesting (but beyond the scope of this paper) to explore a possible interrelationship between the higher-order Noether symmetries and the 'higher-order commutativity conditions' discussed by Sarlet (1980), and of which (A14) is an example.

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